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NOTE

Numerical Iteration of an Elliptic Mixed Boundary-Value Problem in a Region with Curved Boundary*

INTRODUCTION

Linear Partial Differential Equations of the elliptic type in a region R with boundary B, are reduced to a system of finite-difference equations,

$$\sum_{i-1}^{N} a_{ij}u_i = b_j\,, \qquad j = 1,..., N, \tag{1}$$

defined over a square mesh with N interior points in R [1]. If R is rectangular such that the mesh points fall on the boundary B, then for the Dirichlet (first boundary value) problem, the b_j are known constants and the matrix a_{ij} is: (*1*) symmetric; (*2*) diagonally dominant; (*3*) positive-definite; and (*4*) irreducible. Young's theory of successive over-relaxation for the point-by-point iteration [2] establishes these four properties of the coefficient matrix to be sufficient for the convergence of the iteration system. Hence over-relaxation method of point iteration is extensively used for numerical solution of the Dirichlet problem in rectangular regions. This method is extended without much difficulty to either the mixed boundary value problems over the rectangular regions or the Dirichlet problem over a region with curved boundary [3].

However, when applied to the mixed boundary-value problem in a region with curved boundary, with the normal derivative specified on that curved boundary, the method seems to fail [3]. This is in spite of the care taken to approximate the normal derivative in such a way as to keep the coefficient matrix at least diagonally dominant.

For a mixed boundary-value problem, the value of the solution function at the points on the boundary at which the normal derivative is being specified is among the unknowns. Hence the system of equations is larger than that of a corresponding Dirichlet problem. Further, it is obvious that the non-symmetry of the coefficient matrix enters through the approximation of the normal derivative on the curved boundary [3].

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It is being proposed that if initially the values of the function are assumed only on the interior points, and those for the boundary points on which the normal derivative is specified are only adjusted in accordance with the normal derivative specification, then the system may be treated as that corresponding to a Dirichlet problem. Hence during any iteration only the values at the interior points are over-relaxed, and then the values for the boundary points are readjusted so as to remain conformed to the original specification. This way it is made sure that all the boundary conditions are being satisfied while iteration is carried only over a system of equations for which the coefficient matrix is symmetric and diagonally dominant, etc.

The procedure outlined above, amounts to replacing Eq. (1) by a corresponding system as:

$$\sum_{i=1}^{N} a_{ij} u_i^{(n)} = b_j^{(n-1)}, \tag{2}$$

where N is the total number of the interior point only, and n is the number of iterations.

The sufficient conditions for the convergence of the iration system with Eq. (2) are not known, but computational experiments were conducted with extremely encouraging results. The rate of convergence for this iteration process is slower than that for the corresponding Dirichlet problem, where the right-hand side in Eq. (2) remains constant. This is perhaps to be expected. However, a definite trend for convergence is indicated by these experiments.

RESULTS

The following two problems were solved on an IBM 1130 digital computer and the numerical solutions were asked to be plotted on a Calcomp plotter. These are given in Figs. 1 and 2, respectively.

Problem (*1*) Diff. Eq.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{3}$$

Region R: $0 \leqslant x \leqslant 1$,
$0 \leqslant y \leqslant 1 - x^2$.

Bound. Cond: $u(x, 0) = 0$,
$u(0, y) = 1$, (4)

and $\dfrac{\partial u}{\partial n}(x, y) = 0$ for $x^2 + y^2 = 1$.

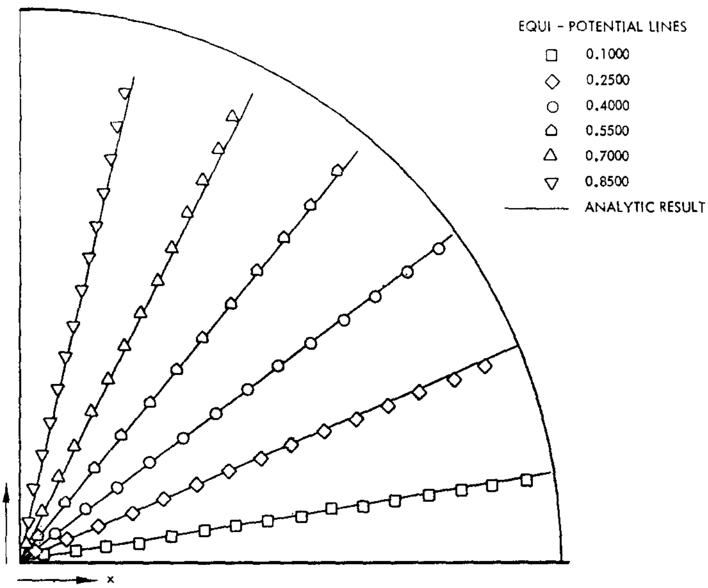

FIG. 1. Solution to Problem 1.

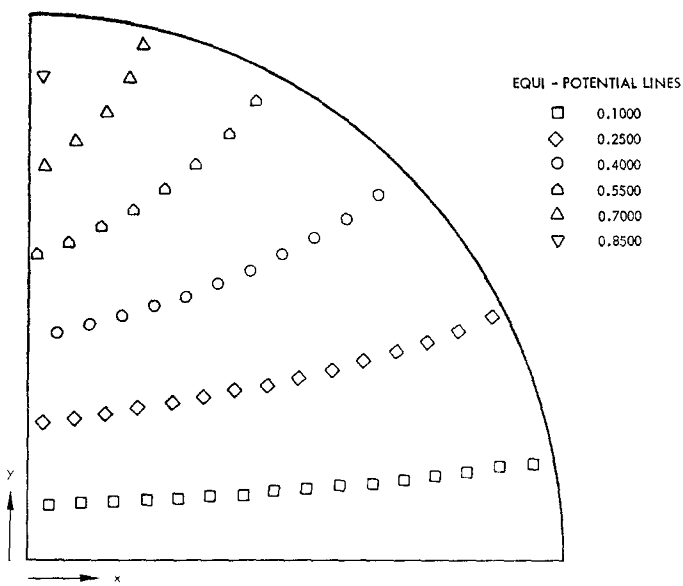

FIG. 2. Solution to Problem 2.

Problem (*2*) Diff. Eq.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{5}$$

Region R: $0 \leqslant x \leqslant 1,$
$0 \leqslant y \leqslant 1 - x^2.$

$$\begin{aligned} \text{Bound. Cond.:} \quad & u(x, 0) = 0, \\ & u(0, y) = y, \\ \text{and} \quad & \frac{\partial u}{\partial n}(x, y) = 0 \quad \text{for} \quad x^2 + y^2 = 1. \end{aligned} \tag{6}$$

Problem (*1*) was chosen, for its analytical solution is available as:

$$u = \frac{2}{\pi} \arctan\left(\frac{y}{x}\right). \tag{7}$$

Equation (7) is shown plotted in Fig. 1 and agrees well with the numercal result.

Concluding Remarks

Based on the numerical experiments of which only two are presented here, it seems reasonable to conclude that digital computers can handle the mixed boundary value problem over regions with curved boundaries. The method has been programmed in FORTRAN language to handle a general linear second-order partial differential equation over a region bounded by the positive coordinate axes and a monotonic-decreasing function representing the third curve. The specifications of either the variable or its normal derivative is possible on any of the bounding curves.

References

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